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CAST ON A LINE SEGMENT IN A POISSON RANDOM FIELD

by

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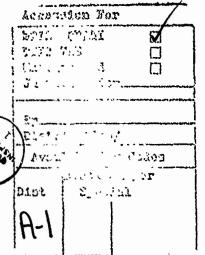Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

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THE DISTRIBUTION OF THE SIZE AND NUMBER OF SHADOWS CAST ON A LINE SEGMENT IN A POISSON RANDOM FIELD

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ABSTRACT

The present paper presents algorithms for the computation of the distributions of the number and lengths of shadows cast on a line segment by random obscurring elements, when there is one source of light (in the origin), and the obscurring elements are realized according to a Poisson random field. These algorithms have applications in problems of line-of-sight, target detection, image processing and others.

Key Words: Visibility Probababilities, Poisson fields, distributions of shadows.

1. Introduction

The following is the physical problem under consideration. One source of light is located at a specified point 0 in space (three dimensional). Consider a straight line $\mathcal{L}$ of distance $r, 0 < r < \infty$, from 0, and a particular segment $\mathcal{C}$ on $\mathcal{L}$. Let $\mathcal{S}$ be a region (strip) in the space of distance u from 0 and distance $r - w$ from $\mathcal{L}$, where $0 < u < w < r$. A countable number of spheres of random radius are randomly dispersed with centers randomly located in $\mathcal{S}$. The location coordinates of the centers of the spheres and their radii constitute a Poisson random field. These random spheres cast shadows in the space. Some of these shadows fall on the line segment $\mathcal{C}$ and obscure it. Due to the randomness of the spheres in $\mathcal{S}$, the number and size of subintervals of $\mathcal{C}$ which are covered by shadows are random variables. The objective of the present paper is to study the distributions of these random variables. Chernoff and Daly (1957) studied for the first time the problem of determining the distribution of the length of a random shadow on a line segment in a Poisson random field in the plane. Explicit algorithms for the determination of this distribution was not given, however, in that paper. In a series of papers, Yadin and Zacks (1982, 1985, 1988) developed formulae for the computation of the probability that

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n points on C are simultaneously visible, moments of random weighted visibility measures and approximations to the distribution of the number of shadows on a line segment (1984). Recently, Yadin and Zacks (1990) applied a functional which was introduced by Chernoff and Daly (1957) to derive the distribution of the length of a shadow. In the present paper we extend these results to obtain also the distribution of the number of shadows on C.

In Section 2 we introduce basic definitions pertaining to the random fields under consideration and to visibility probabilities. In Section 3 we present the distributions of the length of visible and of shadowed subintervals of C. Section 4 is devoted to the recursive determination of the probability function of the number of shadowed subintervals of C. In Section 5 we present numerical examples for two dimensional and three dimensional models. The present study, as well as the above referenced ones of Yadin and Zacks, were motivated by "lines-of-sight" military problems, and the results had been applied by operational analysts dealing with simulated war games. It is conceivable that the methods and results of these studies will be applicable in the fields of image processing, when portions of the image of an object under consideration are obscurred by randomly dispersed objects in the field of vision.

2. The Random Field Model And Visibility Probabilities.

A countable number of spheres (disks) are dispersed so that their centers are points in a region (strip) S of distance $u, 0 < u < \infty$, from the origin, 0. Let C denote a random vector whose components are the coordinates of the center of a random sphere (disk) and let $Y, a \leq Y \leq b, 0 < a < b < r - w$, denote the random radius of such a sphere. Let $S^* = S \times [a, b]$ and B^* the Borel σ-field on S^*. Let $\{(C_i, Y_i), i = 1, 2, \ldots\}$ represent a countable sequence of random disks. The random vectors $(C_i, Y_i), i = 1, 2, \ldots$ are independent and identically distributed, having a c.d.f. $H(c, y)$.

Let $\{B_1, \ldots, B_m\}, m = 2, 3, \ldots$ be any partition of S^*, where $B_j \in B^*(j = 1, \ldots, m)$ and let $N\{Bj\}, j = 1, \ldots, m$, be the number of spheres (disks) having centers in B_j. The random field is called *Poisson* if $N(B_j)$ has a Poisson distribution with mean

$$(2.1) \qquad \mu_j = \lambda \int_{B_j} dH(c, y) \quad , j = 1, \ldots, m, 0 < \lambda < \infty$$

and $N\{B_j\}$ are mutually independent. Let C be a segment on the line $\mathcal{L}$. A point on $\mathcal{L}$ whose distance from 0 is r, is denoted by P_0. A point on $\mathcal{L}$ whose distance from P_0 is $|x|$ is denoted by $P_x, -\infty < x < \infty$. x is the coordinate of P_x on $\mathcal{L}$. A point $P_x \in \mathcal{L}$ is called *visible* from 0, if the ray $\mathbb{R}_x$ from 0 through P_x is not intersected by random disks. Let $\lambda K_{\pm}(x, t)$ denote the expected number of random disks centered in S,

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between the rays $\mathbb{R}_t$ and $\mathbb{R}_{t\pm t}$, which do not intersect $\mathbb{R}_x, 0 \leq t < \infty$. Explicit formulae for $K_{\pm}(x,t)$ will be given later for some special cases. Without loss of generality, assume that the intersection of S with the plane, $\mathcal{L}^*$, containing 0 and $\mathcal{L}$ is a strip bounded by two parallel lines, $\mathcal{U}$ and $\mathcal{W}$, of distance u, and w from $0; 0 < u < w < r < \infty$.

Assume that the segment of interest on $\mathcal{L}, \mathcal{C}$, is an interval whose points $\mathbf{P}_x$ have coordinates $x \in [L, U]$. Let $L^* < L$ and $U^* > U$ and let C^* be a region in S whose intersection with the plane $\mathcal{L}^*$ is the trapez bounded by $\mathbb{R}_{L^*}$ and $\mathbb{R}_{u^*}$. One can verify that the probability that $\mathbf{P}_x$ is visible is

(2.2) $$\psi(x) = \exp\{-[\mu\{C^*\} -)K_{-}(x, x - L^*) - \lambda K_{+}(x, U^* - x)]\}.$$

where $\mu\{C^*\}$ is the expected number of spheres (disks) having centers at C^*. Formulae for the simultaneous visibility of n points in $\mathcal{C}$ an be found in Yadin and Zacks (1985, 1988).

3. Distributions Of Length of Visible And Of Shadowed Segments.

3.1 Distributions Of The Length Of Visible Segments.

In the present section we derive a formula for the conditional c.d.f. of the length of a visible segment to the r.h.s. of $\mathbf{P}_x$, given that $\mathbf{P}_x$ is visible.

Let $I(x)$ be an indicator function which assumes the value 1 if $\mathbf{P}_x$ is visible, and the value zero otherwise.

Let $L(x)$ be the length of the visible setment of $\mathcal{C}$ to the r.h.s. of $\mathbf{P}_x$, i.e.,

(3.1) $$L(x) = \inf\{y : y \geq x, \prod_{x < u \leq y} I(u) = 1\} - x.$$

We derive here the formula for

$$V(l \mid x) = P\{L(x) \leq l | I(x) = 1\}.$$
(3.2) $$= 1 - P\{l(x) > l | I(x) = 1\}.$$

Let C^* be the set of points in S, which was defined in the previous section. We derive the formula of $V(l|x)$, for $L < x < U$, and $0 \leq l \leq u - x$.

Let $C_{-}(x)$ be the set whose intersection with the plane $\mathcal{L}^*$ is bounded by $\mathcal{U}, \mathcal{W}$ and the rays $\mathbb{R}_{L^*}$ and $\mathbb{R}_x$. Let $C(x, l)$ be the set whose intersection with $\mathcal{L}^*$ is bounded by $\mathcal{U}, \mathcal{W}$ and the rays $\mathbb{R}_x, \mathbb{R}_{x+l}$; and $C_{+}(l + x)$ the set whose intersections with $\mathcal{L}^*$ is bounded by $\mathcal{U}, \mathcal{W}, \mathbb{R}_{l+x}$ and $\mathbb{R}_{U^*}$. Notice that $C^* = C_{-}(x) \cup C(x, l) \cup C_{+}(l + x)$. As before, we denote

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by $\mu\{C\}$ the expected number of disks having centers at the set C. Accordingly,

$$P\{L(x) > l, I(x) = 1\} = \exp\{-[\mu\{C_-(x)\} - \lambda K_-(x, x - L^*)] - \mu\{C(x,l)\}$$
$$- [\mu\{C_+(l+x)\} - \lambda K_+(l+x, U^* - l - x)]\}$$
$$(3.3) \qquad = \exp\{-\mu\{C^*\} + \lambda[K_-(x, x - L^*) + K_+(l+x, U^* - l - x)]\}.$$

Dividing (3.3) by (2.2) we obtain

$$(3.4) \qquad P\{L(x) > l | I(x) = 1\} = \exp\{-\lambda[K_+(x, U^* - x) - K_+(l+x, U^* - l - x)]\}.$$

3.2 The Distribution of Shadow Length.

We have denoted by $U(x)$ the right hand limit of the shadow on C to the r.h.s. of $\mathbf{P}_x$. Let $D(u|x)$ denote the conditional c.d.f. of $U(x)$, given that the shadow starts at $\mathbf{P}_x$.

Consider the rays $\mathbb{R}_x$ and $\mathbb{R}_y$ for $y > x$. Let $N(x,y)$ denote the number of disks centered in $\mathcal{S}$, which intersect both $\mathbb{R}_x$ and $\mathbb{R}_y$. Following Chernoff and Daly (1957), we define the functional

$$(3.5) \qquad T(x) = \sup\{y : N(x,y) \geq 1\}.$$

Furthermore, let $T^{i+1}(x) = T(T^i(x)), i = 0, 1, \cdots$ where $T^0(x) = x$. Obviously, $T^{i+1}(x) \geq T^i(x)$, for all $i \geq 0$, and therefore $U(x) = \lim_{i \to \infty} T^i(x)$. $U(x) - x$ is the length of the shadow to the r.h.s. of $\mathbf{P}_x$. We derive first the c.d.f. of $T(x)$. Clearly, $\{T(x) > t\} = \{N(x,t) \geq 1\}$. Thus,

$$(3.6) \qquad P\{T(x) \leq t\} = P\{N(x,t) = 0\} = \exp\{-\mu(x,t)\},$$

where $\mu(x,t) = E\{N(x,t)\}$ is given by

$$(3.7) \quad \mu(x,t) = \mu\{C^*\} - \lambda K_+(x, U^* - x) - \lambda K_-(t, t - L^*) + \lambda K_+(x, \tilde{t} - x) + \lambda K_-(t, t - \tilde{t}),$$

and where $\tilde{t}$ is the coordinate of the bisector between $\mathbb{R}_x$ and $\mathbb{R}_t$. Notice that, since $K_+(x,0) = K_-(x,0) = 0$ for all x,

$$\mu(x,x) = \lim_{t \downarrow x} \mu(x,t)$$
$$(3.8) \qquad = \mu\{C^*\} - \lambda K_+(x, U^* - x) - \lambda K_-(x, x - L^*).$$

Hence,

$$(3.9) \qquad \lim_{t \downarrow x} P\{T(x) \leq t\} = \psi(x),$$

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which is the probability that $\mathbf{P}_x$ is visible. Thus, the c.d.f. of $T(\cdot x)$, $H(t;x)$ is zero for $t < x$, it has a jump point at x, $H(x;x) = \psi(x)$, and is absolutely continuous at $t > x$. This property is inherited by the c.d.f. of $T^n(x)$, $H_n(t;x)$. We provide now the recursive relationship between $H_n(t;x)$ and $H_{n-1}(t;x)$. Introduce the bivariate distribution

$$G_n(t_1, t_2; x) = P\{T^{n-1}(x) \le t_1, T^n(x) \le t_2\}.$$

Since $\{T^n(x) \le t\} \subset \{T^{n-1}(x) \le t\}$,

$$H_n(t;x) = P\{T^n(x) \le t\}$$

(3.10)
$$= G_n(t^*, t; x), \quad \text{all } t^* \ge t.$$

For $x < z < y < t$,

(3.11) $\quad P\{T^n(x) \le t | T^{n-2}(x) = z, T^{n-1}(x) = y\} = \exp\{-[\mu(y,t) - \mu(z,t)]\}.$

Indeed, given that $\{T^{n-2}(x) = z, T^{n-1}(x) = y\}$, $\{T^n(x) > t\}$ if and only if, there exists at least one disk which intersects $\mathbb{R}_y$ and $\mathbb{R}_t$, but *does not* intersect $\mathbb{R}_z$. Hence,

(3.12) $\quad G_n(t_1, t_2; x) = \int_x^{t_1} \int_z^{t_1} \exp\{-[\mu(u, t_2) - \mu(z, t_2)]\} dG_{n-1}(z, u; x).$

These bivariate c.d.f. can be determined recursively, starting with $G_1(t_1, t_2; x) = H(t_2; x)$ for all $t_1 \le t_2$. Moreover,

(3.13) $\quad G_2(t_1, t_2; x) = \int_x^{t_1} e^{-\mu(u, t_2)} \left(\int_x^u e^{\mu(z, t_2)} dz \right) dH(u; x).$

Finally, since $H_{n+1}(t;x) \le H_n(t;x)$ for each $t \ge x$ and all $n = 1, 2, \cdots$ the c.d.f. of $U(x)$ is

(3.14) $\quad P\{U(x) \le t\} = \lim_{n \to \infty} H_n(t; x).$

Thus, $P\{U(x) \le t\} = 0$ for all $t < x$, and $\lim_{t \downarrow x} P\{U(x) \le t\} = \psi(x)$. The conditional c.d.f. of $U(x)$, given $\{I(x) = 0\}$ is

(3.15) $\quad P\{U(x) \le t | I(x) = 0\} = \begin{cases} \dfrac{P\{U(x) \le t\} - \psi(x)}{1 - \psi(x)}, & \text{for } t \ge x \\ 0, & \text{for } t < x \end{cases}.$

We are interested, however, in the conditional c.d.f. $D(u|x)$, where $\mathbf{P}_x$ is the first point (the left hand limit) of the random segment in shadow.

Simple geometric considerations yield that the length of a random shadow cast by a *single* sphere, having left hand limit at P_x, which intersects $\mathcal{L}^*$ with a disk with center on a line parallel to $\mathcal{U}$ at a distance h from O, and disk radius z, is

$$(3.16) \qquad \tilde{U}(x,h,z) = r \, \tan\left(2\sin^{-1}\left(\frac{z}{\sqrt{h^2 + x_c^2}}\right) + \tan^{-1}\left(\frac{x}{r}\right)\right) - x$$

where (x_c, h) are the coordinates of the center of the disk, with

$$(3.17) \qquad x_c = \frac{x}{r}h + z\left(1 + \left(\frac{x}{r}\right)^2\right)^{1/2}.$$

Thus, if $a \le Z \le b$ w.p. 1, the minimal length of shadow starting at x is
(3.18)

$$\tilde{U}_m(x) = r \tan\left(2\sin^{-1}\left(\left(\frac{a}{w}\right)\left(1 + \left(\frac{x}{r} + \frac{a}{w}\left(1 + \left(\frac{x}{r}\right)^2\right)^{1/2}\right)^2\right)^{-1/2}\right) + \tan^{-1}\left(\frac{x}{r}\right)\right) - x.$$

Finally, since a shadow starting at P_x ends at point $U(x) \ge \tilde{u}_m(x) + x$,

$$(3.19) \qquad D(u \mid x) = \frac{P\{U(x) \le u\} - P\{U(x) \le \tilde{u}_m(x) + x\}}{1 - P\{U(x) \le \tilde{u}_m(x) + x\}}, \quad \text{for } u \ge x.$$

4. The Distribution of The Number of Shadows

Let $J(x,y)$, for $L^* \le x < y \le U^*$, denote the number of shadows (invisible segments) on the interval (x,y) on $\mathcal{C}$. Let $P_j(x,y) = P\{J(x,y) = j \mid I(x) = 1\}$. Obviously,

$$(4.1) \qquad P_0(x,y) = 1 - V(y - x \mid x)$$

The stochastic process $\{I(X(t)), L^* < t < U^*\}$ is a regenerative process. A cycle in this process is a pair of visible segment followed by a shadowed segment. Let $\zeta(a)$ denote the length of a cycle starting at P_a. The distribution of $\zeta(a)$ is given by the convolution

$$(4.2) \qquad F(x \mid a) = \int_0^{x-a} D(x - a \mid t)dV(t \mid a).$$

It follows immediately that the probability function $P_j(x,y)$ satisfies the recursive equation:

$$(4.3) \qquad P_{j+1}(x,y) = \int_0^{y-x} P_j(t + x \mid y)dF(t \mid x), j \ge 1$$

and

$$P_1(x,y) = \int_0^{y-x} [1 - V(y - t - x \mid t + x)]dF(t \mid x)$$

$$\text{(4.4)} \qquad + \int_0^{y-x} [1 - D(y \mid t + x)]dV(t \mid x).$$

In the following section we present an example in which the various distributions are approximated numerically.

5. An Example

In the present example we restrict attention to the visibility problem in the plane. The region S is a strip with boundaries U and W, parallel to $\mathcal{L}$. C is an interval on $\mathcal{L}$, i.e., $C = [L, U]$ and $C^* = [L^*, U^*], -\infty < L^* < L < U < U^* < \infty$. The distances of U, W and $\mathcal{L}$ from 0 are u, w, r, respectively, $0 < u < w < r$. The Poisson field is a standard-uniform one, i.e., centers of disks, $c_1, c_2, \ldots$ are uniformly and independently distributed in S, and the radii of disks are independent of their center locations and have an identical uniform distribution on (a, b). No disk in S can intersect 0 or $\mathcal{L}$, i.e., $(b - a) < u$ and $r - w > (b - a)$.

5.1 The Functions $K_\pm(x, t)$ in the Standard-Uniform Case, With Uniform Distribution of radii on (a, b).

Let $K_+(x, t, z)$ denote the area of the set bounded by the line $\mathcal{L}_z^+$, the ray $\mathbb{R}_{x+t}, t \geq 0$, and the lines U and W; $\mathcal{L}_z^+$ is the line parallel to $\mathbb{R}_x$, on its r.h.s., of distance z from it. This is the set of all disk centers between $\mathbb{R}_x$ and $\mathbb{R}_{x+t}$, of radius $Z = z$, which do not intersect $\mathbb{R}_x$. In order to simplify notation, we assume that $w = r$. In actual computations we substitute xw/r and tw/r for x and t in the formulae given below. Let $d = (x^2 + w^2)^{1/2}$. Simple geometrical considerations yield:

$$K_+(x, t, z) = I\left\{\frac{zd}{t} < u\right\}\left[\frac{w^2 - u^2}{2w}t - 2\frac{zd}{w + u}\right]$$

$$\text{(5.1)} \qquad + I\left\{u \leq \frac{zd}{t} < w\right\}\left[\frac{1}{2tw}(tw - zd)^2\right]$$

where $I\{A\}$ is the indicator set function, which asumes the value 1 if A is true, and the value 0 otherwise.

Notice that $K_+(x, t, z)$ depends on x only via x^2. Symmetry implies that $K_-(-x, t, z) = K_+(x, t, z) = K_+(-x, t, z)$ for all $-\infty < x < \infty$. Hence, $K_+(x, t) = K_-(x, t)$ and we delete the $\pm$ subscript of K. Finally, $K(x, t) = E\{K(x, t, Z)\}$ with respect to the uniform distribution of Z over (a, b). Let $x_1 = tu/d$ and $x_2 = tw/d$. The function $K(x, t)$ assumes the following forms:

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(i) If $b < x_1$,

$$(5.2) \qquad K(x,t) = \frac{w^2 - u^2}{2w}\left(t - \frac{d}{u+w}(a+b)\right).$$

(ii) If $a < x_1 < b \le x_2$

$$K(x,t) = \frac{w^2 - u^2}{2w}\left(t \cdot \frac{x_1 - a}{b-a} - \frac{d}{u+w}\cdot\frac{1}{b-a}(x_1^2 - a^2)\right)$$
$$(5.3) \qquad + \frac{1}{2tw}\left(t^2 w^2 \frac{b - x_1}{b-a} - tw\frac{d}{b-a}(b^2 - x_1^2) + \frac{d^2}{3(b-a)}(b^3 - x_1^3)\right).$$

(iii) If $a < x_1 < x_2 \le b$

$$K(x,t) = \frac{w^2 - u^2}{2w}\left(t\frac{x_1 - a}{b-a} - \frac{d}{u+w}\frac{1}{b-a}(x_1^2 - a^2)\right)$$
$$(5.4) \qquad + \frac{1}{2tw}\left(t^2 w^2 \frac{x_2 - x_1}{b-a} - tw\frac{d}{b-a}(x_2^2 - x_1^2) + \frac{d^2}{3(b-a)}(x_2^3 - x_1^3)\right).$$

(iv) If $x_1 \le a < b \le x_2$,

$$(5.5) \qquad K(x,t) = \frac{tw}{2} - \frac{d}{2}(a+b) + \frac{d^2(a^2 + ab + b^2)}{6tw}.$$

(v) If $x_1 \le a < x_2 \le b$

$$(5.6) \qquad K(x,t) = \frac{tw}{2}\frac{x_2 - a}{b-a} - \frac{d}{2(b-a)}(x_2^2 - a^2) + \frac{d^2}{6tw(b-a)}(x_2^3 - a^3).$$

(vi) If $x_2 < a$

$$(5.7) \qquad K(x,t) = 0$$

5.2 Algorithms For Discrete Approximations.

In the present section we consider discrete approximations to the functions $H_n(t;x), G_n$ $(t_1, t_2; x), n \ge 2$.

For a given integer, M, partition the interval (x,y) to M subintervals. Accordingly, let $\delta = (y-x)/M, t_0 = x$ and $t_j = t_0 + j\delta, j = 0, 1, \cdots, M$.

For $i = 0, \cdots, M$, let

$$(5.8) \qquad \hat{H}_1(i) = H(t_i; t_0) = \exp\{-\mu(t_0, t_i)\}.$$

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We compute next the function $\hat{G}_2(i,j), i,j = 0, \cdots, M$; which is a discrete approximation to (3.13). For $i = 0, \cdots, M$ and $j = i, \cdots, M$,

$$(5.9) \qquad \hat{G}_2(i,j) = \sum_{k=0}^{i} \exp\{-(\mu(t_k, t_j) - \mu(t_0, t_j))\} \cdot [\hat{H}_1(k) - \hat{H}_1(k-1)],$$

where $\hat{H}_1(-1) = 0$. Notice that $\hat{G}_2(0,j) = \hat{H}_1(0)$ for all $j = 0, 1, \cdots, M$; and for $i \geq 1, j \geq i$

$$(5.10) \qquad \hat{G}_2(i,j) = \hat{H}_1(0) + \exp\{\mu(t_0, t_j)\} \cdot \sum_{k=1}^{i} \exp\{-\mu(t_k, t_j)\}[\hat{H}_1(k) - \hat{H}_1(k-1)].$$

Moreover,

$$(5.11) \qquad\qquad \hat{G}_2(i,j) = \hat{G}_2(j,j) \text{ for all } i > j.$$

Next, compute recursively, for every $n \geq 3, i = 1, \cdots, M, j = i, \cdots, M$

$$\hat{G}_n(i,j) = \sum_{k=0}^{i}\sum_{l=k}^{i} \exp\{-(\mu(t_l, t_j) - \mu(t_k, t_j))\}[\hat{G}_{n-1}(k,l) - \hat{G}_{n-1}(k-1,l)$$

$$(5.12) \qquad\qquad - \hat{G}_{n-1}(k, l-1) + \hat{G}_{n-1}(k-1, l-1)],$$

and for $i > j$

$$(5.13) \qquad\qquad \hat{G}_n(i,j) = \hat{G}_n(j,j).$$

For $i = 0, \hat{G}_n(0,j) = \hat{G}_{n-1}(0,j) = \hat{H}_1(0), j = 0, \cdots, M$. After computing these functions we determine $\hat{H}_n(j) = \hat{G}_n(j,j), j = 0, 1, \cdots, M$. $\hat{H}_n(j)$ is the discrete approximation to the c.d.f. on $T^n(x)$, namely $H_n(t; x)$; i.e., $H_n(t_j; x) \approx \hat{H}_n(j)$.

In Table 5.1 we present numerical results obtained by applying this algorithm to the following special case.

5.3 Numerical Example.

We compute the numerical example for Table 5.1 with the following geometrical parameters: $r = 100[m], u = 40[m], w = 60[m], a = 1[m], b = 2.5[m], x = 10[m], L^* = -100[m], U^* = 100[m]$. We present in the tables the values of $\hat{H}_n(j), n = 1, 2, 3, j = 0, \cdots, 20$, when $\delta = 1[m]$.

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Table 5.1. *Values of $\hat{H}_n(j)$ for two values of λ.*

j	$\lambda = 0.02[1/m^2]$			$\lambda = 0.2[1/m^2]$		
	$\hat{H}_1(j)$	$\hat{H}_2(j)$	$\hat{H}_3(j)$	$\hat{H}_1(j)$	$\hat{H}_2(j)$	$\hat{H}_3(j)$
0	0.4914	0.4914	0.4914	0.0000	0.0000	0.0000
1	0.5434	0.5434	0.5434	0.0000	0.0000	0.0000
2	0.6009	0.6008	0.6008	0.0000	0.0000	0.0000
3	0.6645	0.6642	0.6442	0.0003	0.0003	0.0003
4	0.7342	0.7336	0.7336	0.0021	0.0020	0.0020
5	0.8039	0.8030	0.8030	0.0130	0.0119	0.0119
6	0.8664	0.8652	0.8652	0.0576	0.0522	0.0522
7	0.9182	0.9170	0.9170	0.1828	0.1663	0.1663
8	0.9569	0.9559	0.9559	0.4164	0.3843	0.3842
9	0.9811	0.9804	0.9804	0.6842	0.6484	0.6482
10	0.9934	0.9930	0.9930	0.8763	0.8507	0.8505
11	0.9985	0.9984	0.9984	0.9707	0.9590	0.9588
12	0.9999	0.9999	0.9999	0.9981	0.9958	0.9957
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

As seen in Table 5.1, the convergence of $\hat{H}_n(j)$ to the c.d.f. of $U(x)$ is quite rapid. We have therefore approximated the c.d.f. $D(u \mid x)$ by the sequence $\hat{D}(j \mid i) = D(t_j \mid t_i), i = 0, 1, \cdots, M, j = i, i+1, \cdots$. The c.d.f. of $\zeta(a), F(x \mid a)$ was computed by numerical convolutions of $D(y \mid t)$ with $V(t \mid x)$. Finally the probability functions $P_j(x, y)$ were computed recursively. In Tables 5.2 we present the functions $P_j(x, y)$ for the interval $(-30, 70)$ for various values of λ. The field parameters are $r = 100, u = 40, w = 60, a = 0, b = 3, L^* = -100, U^* = 100$ with $\delta = 1$. We see in Table 5.2 that the number of shadows on $(-30, 70)$ is stochastically increasing with λ up to $\lambda = 0.012$. For $\lambda = 0.024$ the number of shadows starts to decrease stochastically. We have here a delicate interplay between length and number of shadows.

Table 5.2. *Probability Distributions of* $J(x, y)$,
for $\lambda = 0.003, 0.006, 0.012, 0.024; x = -30, y = 70$.

j	λ			
	0.003	0.006	0.012	0.024
0	0.0490	0.0024	0.0000	0.0000
1	0.1841	0.0270	0.0010	0.0016
2	0.2925	0.1147	0.0277	0.1863
3	0.2598	0.2414	0.1600	0.4639
4	0.1444	0.2840	0.3156	0.2713
5	0.0535	0.2020	0.2891	0.0670
6	0.0137	0.0925	0.1476	0.0091
7	0.0025	0.0287	0.0471	0.0008
8	0.0003	0.0063	0.0001	0.0000
9	0.0000	0.0010	0.0016	0.0000
10	0.0000	0.0001	0.0002	0.0000

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The present paper presents algorithms for the computation of the distributions of the number and lengths of shadows cast on a line segment by random obscurring elements, when there is one course of light (in the origin), and the obscurring elements are realized according to a Poisson random field. These algorithms have applications in problems of line-of-sight, target detection, image processing and others.

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